

# Ehrenfest Theorem in Precanonical Quantization of Fields and Gravity

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We discuss a generalization of the Ehrenfest theorem to the recently proposed precanonical quantization of vielbein gravity which proceeds from a space-time symmetric generalization of the Hamiltonian formalism to field theory. Classical Einstein-Palatini equations are derived as equations of expectation values of precanonical quantum operators. The preceding consideration of an interacting scalar field theory on curved space-time shows how the classical field equations emerge from the results of precanonical quantization as the equations of expectation values of the corresponding quantum operators. It also allows us to identify the connection term in the covariant generalization of the precanonical Schrödinger equation with the spin connection.

*Keywords:* Quantum field theory in curved space-time; quantum gravity; precanonical quantization; De Donder-Weyl theory; vielbein gravity; Clifford algebra; Ehrenfest theorem; classical limit.

## 1. Introduction

In our previous papers<sup>1,2</sup> we have been exploring the potential of the De Donder-Weyl (DW) space-time symmetric generalization of the Hamiltonian formalism to field theory<sup>3</sup> as a basis of field quantization. The resulting *precanonical quantization* is based on the mathematical structures of the DW Hamiltonian formalism, such as the polysymplectic  $(n+1)$ -form (generalizing the symplectic 2-form in mechanics to field theories in  $n$  dimensions) and the Poisson-Gerstenhaber algebra of dynamical variables represented by a certain class of differential forms (generalizing the Poisson algebra of functions/functionals on the phase space in the canonical formalism of mechanics/field theory).<sup>4</sup> It leads to the formulation of a quantum theory of fields in terms of Clifford algebra-valued wave functions and operators on the finite dimensional space of field coordinates and space-time coordinates,<sup>1,2</sup> which generalizes the configuration space in mechanics to field theory without the usual splitting to space and time. As the space-time coordinates enter the theory on the equal footing - as a multidimensional analog of the time variable - the corresponding space-time Clifford (Dirac) algebra generalizes the algebra of complex numbers in quantum mechanics (and its generalization to the continually infinite number of degrees of freedom, the standard QFT) to the present description of fields as “multi-temporal” systems. The standard QFT in the functional Schrödinger representation<sup>5</sup> was shown to be derivable from precanonical quantization in the limit of the vanishing “elementary volume”  $\frac{1}{\varkappa}$ ,<sup>6</sup> which appears in the formalism of precanonical quantization when the dynamical variables represented by differential forms, which are infinitesimal volume elements, are represented by dimensionless elements of Clifford algebra.

Here, we first discuss the precanonical quantization of interacting scalar fields on

the curved space-time background and show how the classical field equations in DW Hamiltonian form are derived from precanonical quantization as the equations of expectation values of quantum operators. Then we briefly outline the precanonical quantization of vielbein general relativity in Palatini formulation and sketch out the derivation of the classical Einstein-Palatini equations in DW Hamiltonian form from the precanonical quantization of gravity as the equations of expectation values of the corresponding operators. That generalizes the Ehrenfest theorem to the precanonical quantization of gravity.<sup>7,9</sup> Further details of the derivation will be presented elsewhere.<sup>8</sup>

The concise presentation here is dictated by the limitations of the proceedings format of this paper. The reader is advised to consult the previous papers by the author for the notation, terminology, concepts, and the formalism underlying the precanonical quantization, which are used here mostly without explanation. A companion paper<sup>10</sup> may serve as a brief introduction.

## 2. Ehrenfest Theorem in curved space-time

Let us consider interacting scalar fields on a curved space-time background  $g^{\mu\nu}(x)$ :

$$\mathfrak{L} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu y^a\partial_\nu y_a - \sqrt{g}V(y), \quad (1)$$

where  $g := |\det g_{\mu\nu}|$  and the parametric dependences from  $x$  are henceforth not written down explicitly. Our purpose here is to present the DW Hamiltonian formulation, the resulting precanonical quantization of the system, and then to show that the latter reproduces the DW Hamiltonian equations as the equations on the expectation values of the corresponding operators.

The polymomenta and the DW Hamiltonian density obtained from (1):

$$\mathfrak{p}_a^\mu = \sqrt{g}g^{\mu\nu}\partial_\nu y_a, \quad \mathfrak{H} = \sqrt{g}H = \frac{1}{2\sqrt{g}}g_{\mu\nu}\mathfrak{p}_a^\mu\mathfrak{p}_a^{\nu} + \sqrt{g}V(y), \quad (2)$$

are densities which parametrically depend on the space-time coordinates  $x$  via  $g^{\mu\nu}(x)$ -s. The DW Hamiltonian form of the Euler-Lagrange equations reads

$$\mathbf{d}_\mu\mathfrak{p}_a^\mu(x) = -\partial_a\mathfrak{H}, \quad \mathbf{d}_\mu y^a(x) = \partial_{\mathfrak{p}_a^\mu}\mathfrak{H}, \quad (3)$$

where  $\mathbf{d}_\mu$  is the total differentiation w.r.t.  $x^\mu$  and  $\partial_a := \partial/\partial y^a$ ,  $\partial_{\mathfrak{p}_a^\mu} := \partial/\partial\mathfrak{p}_a^\mu$ .

Precanonical quantization of this system leads to the representations

$$\hat{\mathfrak{p}}_a^\mu = -i\hbar\kappa\sqrt{g}\gamma^\mu\partial_a, \quad \hat{H} = -\frac{1}{2}\hbar^2\kappa^2\partial_a\partial^a + V(y), \quad (4)$$

where the curved-space  $\gamma^\mu$ -matrices are  $x$ -dependent:  $\gamma^\mu\gamma^\mu + \gamma^\mu\gamma^\mu = 2g^{\mu\mu}$ , while the DW Hamiltonian operator turns out to be independent from  $x$ -s. The curved space-time covariant generalization of the precanonical Schrödinger equation takes the form

$$i\hbar\kappa\gamma^\mu\nabla_\mu\Psi(y, x) = \hat{H}\Psi(y, x), \quad (5)$$

where  $\nabla_\mu := \partial_\mu + \omega_\mu$  with the connection  $\omega_\mu$  is a covariant derivative of Clifford algebra-valued wave functions. For the conjugate wave function  $\bar{\Psi} := \bar{\gamma}^0 \Psi^\dagger \bar{\gamma}^0$  we obtain:

$$i\hbar\kappa\bar{\Psi}(\overleftarrow{\partial}_\mu + \bar{\omega}_\mu)\gamma^\mu = -\hat{H}\bar{\Psi}, \quad (6)$$

where  $\bar{\Psi} := \bar{\gamma}^0 \Psi^\dagger \bar{\gamma}^0$ ,  $\bar{\gamma}^I$  ( $I = 0, \dots, n-1$ ) are flat-space Dirac matrices, such that  $\bar{\gamma}^I \bar{\gamma}^J + \bar{\gamma}^J \bar{\gamma}^I = 2\eta^{IJ}$ ,  $\eta^{IJ}$  is the fiducial flat-space metric,  $\bar{\omega}_\mu := \bar{\gamma}^0 \omega_\mu^\dagger \bar{\gamma}^0$ , and a generalized Hermiticity of  $\hat{H}$  is assumed:  $\hat{H} = \overline{\hat{H}} := \bar{\gamma}^0 \hat{H}^\dagger \bar{\gamma}^0$ .

From (5) and (6) we obtain the covariant conservation law

$$\mathbf{d}_\mu \int dy \text{Tr}(\bar{\Psi} \sqrt{g} \gamma^\mu \Psi) = 0, \quad (7)$$

provided  $\omega_\mu$  satisfies the known property of the spin connection  $\omega_\mu = \frac{1}{4} \omega_\mu^{IJ} \bar{\gamma}_{IJ}$ :<sup>11</sup>

$$\partial_\mu(\sqrt{g} \gamma^\mu) = \sqrt{g} \bar{\omega}_\mu \gamma^\mu + \sqrt{g} \gamma^\mu \omega_\mu. \quad (8)$$

Now, from (4), (5), (6) and (8), we obtain:

$$\begin{aligned} \mathbf{d}_\mu \langle \hat{\mathbf{p}}_a^\mu \rangle &= -i\hbar\kappa \mathbf{d}_\mu \int dy \text{Tr}(\bar{\Psi} \sqrt{g} \gamma^\mu \partial_a \Psi) = -i\hbar\kappa \int dy \text{Tr}(\partial_\mu \bar{\Psi} \sqrt{g} \gamma^\mu \partial_a \Psi \\ &+ \bar{\Psi} \partial_a \sqrt{g} \gamma^\mu \partial_\mu \Psi + \bar{\Psi} \partial_\mu (\sqrt{g} \gamma^\mu) \partial_a \Psi) = - \int dy \text{Tr}(\bar{\Psi} [\partial_a, \hat{\mathbf{S}}] \Psi) = -\langle \partial_a \hat{\mathbf{S}} \rangle. \end{aligned}$$

Therefore, the first DW Hamiltonian equation in (3) is fulfilled on average.

Next, we note that (3b) can be reformulated in terms of the covariant derivative and an  $x$ -dependent contravariant  $(n-1)$ -volume element  $\varpi^\mu := g^{\mu\nu} \varpi_\nu$ , where  $\varpi_\nu := \iota_{\partial_\nu} (dx^0 \wedge \dots \wedge dx^{n-1})$ :

$$\nabla_\mu (y^a \varpi^\mu) = \mathbf{d}_\mu (y^a \varpi^\mu) + \frac{1}{2} y^a \partial_\mu (\ln g) \varpi^\mu = \partial_{\mathbf{p}_a^\nu} \hat{\mathbf{S}} \varpi^\mu. \quad (9)$$

Using the representation  $\hat{\varpi}^\mu(x) = \frac{1}{\kappa} \gamma^\mu(x)$ , and setting here  $\hbar=1$ ,  $\kappa=1$ , we obtain

$$\begin{aligned} \mathbf{d}_\mu \langle \widehat{y^a \varpi^\mu} \rangle &= \int dy \text{Tr}(\partial_\mu \bar{\Psi} y^a \gamma^\mu \Psi + \bar{\Psi} y^a \gamma^\mu \partial_\mu \Psi + \bar{\Psi} y^a (\partial_\mu \gamma^\mu) \Psi) \\ &= \int dy \text{Tr}(\bar{\Psi} (i \overleftarrow{\hat{H}} - \bar{\omega}_\mu \gamma^\mu) y^a \Psi - \bar{\Psi} y^a (i \hat{H} + i \gamma^\mu \omega_\mu) \Psi + \hbar\kappa \bar{\Psi} y^a (i \partial_\mu \gamma^\mu) \Psi) \\ &= \int dy \text{Tr}(\bar{\Psi} [\hat{H}, i y^a] \Psi + i \bar{\Psi} (\partial_\mu \gamma^\mu - \bar{\omega}_\mu \gamma^\mu - \gamma^\mu \omega_\mu) y^a \Psi) \\ &= \langle -i \partial^a \rangle - \frac{1}{2} \langle y^a \gamma^\mu \rangle \partial_\mu (\ln g) = \langle \partial_{\mathbf{p}_a^\nu} \hat{\mathbf{S}} \varpi^\nu \rangle - \frac{1}{2} \langle y^a \gamma^\mu \rangle \partial_\mu (\ln g), \end{aligned} \quad (10)$$

thus reproducing on average the second DW Hamiltonian equation in (3) in the form (9), if  $\omega_\mu$  satisfies the condition (8) and hence can be identified with the spin connection.

Therefore, the analog of the Ehrenfest theorem for the precanonically quantized scalar fields on curved space-time is satisfied as the consequence of (i) the covariant precanonical Schrödinger equation (5) and its conjugate (6), (ii) the precanonical representation of operators (4), (iii) the definition of the scalar product related to the conservation law (7), and (iv) the property (8) of the connection term in (5), which allows us to identify it with the spin connection (the Fock-Ivanenko coefficients).

### 3. Precanonical quantization of vielbein gravity and the Ehrenfest Theorem

Here we follow our earlier work<sup>12</sup> (cf. Refs. 13 for an earlier work using the metric formulation). The Einstein-Palatini Lagrangian density with the cosmological term

$$\mathcal{L} = \frac{1}{\kappa} \epsilon e_I^{[\alpha} e_J^{\beta]} (\partial_\alpha \omega_\beta^{IJ} + \omega_\alpha^{IK} \omega_{\beta K}^J) + \frac{1}{\kappa} \Lambda \epsilon, \quad \epsilon := (\det ||e_I^\mu||)^{-1}, \quad (11)$$

treats the vielbein components  $e_I^\mu$  and the spin connection coefficients  $\omega_\alpha^{IJ}$  as independent field variables. The DW Hamiltonian formulation leads to the momenta  $\mathbf{p}_e^\alpha$  and  $\mathbf{p}_\omega^\alpha$ , and the DW Hamiltonian density  $\mathfrak{H} =: \epsilon H$  derived from (11):

$$(a): \mathbf{p}_{e_I^\beta}^\alpha \approx 0, \quad (b): \mathbf{p}_{\omega_\beta^{IJ}}^\alpha \approx \frac{1}{\kappa} \epsilon e_I^{[\alpha} e_J^{\beta]}, \quad (c): \mathfrak{H} = -\frac{1}{\kappa} \epsilon e_I^{[\alpha} e_J^{\beta]} \omega_\alpha^{IK} \omega_{\beta K}^J - \frac{1}{\kappa} \Lambda \epsilon. \quad (12)$$

The primary constraints (12a,b) are second-class, because the brackets of their associated  $(n-1)$ -forms  $\mathfrak{C}_{e_I^\beta} := \mathbf{p}_{e_I^\beta}^\alpha \varpi_\alpha$ ,  $\mathfrak{C}_{\omega_\beta^{IJ}} := (\mathbf{p}_{\omega_\beta^{IJ}}^\alpha - \frac{1}{\kappa} \epsilon e_I^{[\alpha} e_J^{\beta]}) \varpi_\alpha$  are not all vanishing:

$$\{\mathfrak{C}_e, \mathfrak{C}_{e'}\} = 0 = \{\mathfrak{C}_\omega, \mathfrak{C}_{\omega'}\}, \quad \{\mathfrak{C}_{e_K^\gamma}, \mathfrak{C}_{\omega_\beta^{IJ}}\} = -\frac{1}{\kappa} \partial_{e_K^\gamma} (\epsilon e_I^{[\alpha} e_J^{\beta]}) \varpi_\alpha =: \mathfrak{C}_{e_K^\gamma \omega_\beta^{IJ}}. \quad (13)$$

**Einstein's equations** are derived from (11) by varying  $\omega$  and  $e$  independently:

$$\delta\omega: \nabla_\alpha (\epsilon e_{[I}^\alpha e_{J]}^\beta) = 0, \quad \delta e: \partial_{e_M^\mu} (\epsilon e_I^{[\alpha} e_J^{\beta]} (\partial_\alpha \omega_\beta^{IJ} + \omega_\alpha^{IK} \omega_{\beta K}^J) + \Lambda \epsilon) = 0. \quad (14)$$

The former equation defines the spin connection in terms of vielbeins, and the latter one is the vacuum Einstein's equation in vielbein formulation. In the DW Hamiltonian formulation these equations can be written in the form:

$$(a): \mathbf{d}_\alpha \mathbf{p}_{\omega_\beta^{IJ}}^\alpha = -\partial_{\omega_\beta^{IJ}} \mathfrak{H}, \quad (b): \mathfrak{C}_{e_K^\gamma \omega_\beta^{IJ}} \mathbf{d}_{[\alpha} \omega_{\beta]}^{IJ} = -\partial_{e_K^\gamma} \mathfrak{H}, \quad (15)$$

where  $\mathfrak{C}_{e\omega} =: \mathfrak{C}_{e\omega}^\alpha \varpi_\alpha$  and the total (on-shell) derivative  $\mathbf{d}$  also implies a restriction to the subspace of constraints (12), e.g.  $\mathbf{d}_\alpha \mathbf{p}_\omega^\alpha = \frac{\partial \mathbf{p}_\omega^\alpha(e)}{\partial e} \frac{\partial e}{\partial x^\alpha}$ . Note that this formulation is inspired by the generalized Dirac analysis<sup>9,12,14</sup> of the DW Hamiltonian system with the constraints (12).

**Quantization** yields the representation of the operators of vielbeins and momenta:  $\hat{e}_I^\beta = -i\hbar \kappa \bar{\gamma}^J \frac{\partial}{\partial \omega_\beta^{IJ}}$ ,  $\hat{\mathbf{p}}_{\omega_\beta^{IJ}}^\alpha = -\hbar^2 \kappa^2 \kappa \hat{\mathbf{e}} \bar{\gamma}^{KL} \frac{\partial}{\partial \omega_{[\alpha}^{KL}} \frac{\partial}{\partial \omega_{\beta]}^{IJ}}$ , which act on Clifford-valued precanonical wave functions  $\Psi = \Psi(\omega_\alpha^{IJ}, x^\mu)$  on the (total space of the) configuration bundle of spin connections over the space-time. For the operator of the DW Hamiltonian density  $\mathfrak{H} =: \epsilon H$  restricted to the surface of constraints  $C$  given by (12):  $(\epsilon H)|_C = -\mathbf{p}_{\omega_\beta^{IJ}}^\alpha \omega_\alpha^{IK} \omega_{\beta K}^J - \frac{1}{\kappa} \Lambda \epsilon$ , we obtain (up to an ordering  $\vdots \dots \vdots$ )

$$\hat{H} = \hbar^2 \kappa^2 \kappa \bar{\gamma}^{IJ} \vdots \omega_{[\alpha}^{KM} \omega_{\beta]M}^L \partial_{\omega_\alpha^{IJ}} \partial_{\omega_{\beta}^{KL}} \vdots - \frac{1}{\kappa} \Lambda. \quad (16)$$

**Precanonical Schrödinger equation** for quantum gravity:  $i\hbar\hat{\mathcal{V}}\Psi = \hat{H}\Psi$ , where  $\hat{\mathcal{V}} := \hat{\gamma}^\mu(\partial_\mu + \frac{1}{4}\omega_{\mu IJ}\hat{\gamma}^{IJ})$  and  $\hat{\gamma}^\mu := \bar{\gamma}^I\hat{e}_I^\mu = -i\hbar\kappa\bar{\gamma}^{IJ}\partial_{\omega_\mu^{IJ}}$  (cf. (5)), now reads

$$\bar{\gamma}^{IJ}\partial_{\omega_\mu^{IJ}}\partial_\mu\Psi + \bar{\gamma}^{IJ}:(\frac{1}{4}\omega_{\mu KL}\bar{\gamma}^{KL} - \omega_{\mu M}{}^K\omega_\beta^{ML}\partial_{\omega_\mu^{KL}})\partial_{\omega_\beta^{IJ}}:\Psi + \lambda\Psi = 0, \quad (17)$$

where  $\lambda := \frac{\Lambda}{(\hbar\kappa\kappa)^2}$  is a dimensionless constant which combines three different scales: cosmological, Planck, and the UV scale  $\kappa$  introduced by precanonical quantization.

The conjugate wave function  $\bar{\Psi} := \bar{\gamma}^0\Psi^\dagger\bar{\gamma}^0$  obeys

$$\partial_{\omega_\mu^{IJ}}\partial_\mu\bar{\Psi}\bar{\gamma}^{IJ} - \bar{\Psi}:(\frac{1}{4}\omega_{\mu KL}\bar{\gamma}^{KL}\bar{\gamma}^{IJ} + \omega_{\mu M}{}^K\omega_\beta^{ML}\overleftarrow{\partial}_{\omega_\beta^{KL}})\overleftarrow{\partial}_{\omega_\mu^{IJ}}:-\lambda\bar{\Psi} = 0. \quad (18)$$

**The scalar product** of precanonical wave functions is given by

$$\langle\Phi|\Psi\rangle := \text{Tr} \int \bar{\Phi}[\widehat{d\omega}]\Psi, \quad [\widehat{d\omega}] = i^{\frac{1}{2}n(n+1)-1}\hat{\mathbf{e}}^{-n(n-1)} \prod_{\mu, I < J} d\omega_\mu^{IJ}, \quad (19)$$

where  $[\widehat{d\omega}]$  is a Misner-like diffeomorphism invariant generalized-Hermitian operator-valued measure on the fibers of the configuration bundle of spin connections over space-time and

$$\hat{\mathbf{e}}^{-1} = \frac{1}{n!}\epsilon^{I_1\dots I_n}\epsilon_{\mu_1\dots\mu_n}\hat{e}_{I_1}^{\mu_1}\dots\hat{e}_{I_n}^{\mu_n} = \frac{(-i)^n}{n!}\bar{\gamma}^*\epsilon^{I_1\dots I_n}\epsilon^{J_1\dots J_n}\epsilon_{\mu_1\dots\mu_n}\partial_{\omega_{\mu_1}^{I_1J_1}}\dots\partial_{\omega_{\mu_n}^{I_nJ_n}}, \quad (20)$$

where  $\bar{\gamma}^* := \bar{\gamma}^1\bar{\gamma}^2\dots\bar{\gamma}^n$ . The numerical factor in (19b) (which was omitted in Refs. 12) comes from the generalized-Hermiticity requirement:  $[\widehat{d\omega}] = \overline{[\widehat{d\omega}]}$ .

The expectation values of operators are calculated as

$$\langle\hat{O}\rangle(x) = \text{Tr} \int \bar{\Psi}(\omega, x):\hat{O}[\widehat{d\omega}]:\Psi(\omega, x). \quad (21)$$

**The conservation law** derived from (17) and its conjugate (18) (setting  $\hbar=1=\kappa$ ):

$$\begin{aligned} \partial_\mu \int \bar{\Psi}\hat{\gamma}^\mu[\widehat{d\omega}]\Psi &= \int \partial_\mu \bar{\Psi}\hat{\gamma}^\mu[\widehat{d\omega}]\Psi + \int \bar{\Psi}\hat{\gamma}^\mu[\widehat{d\omega}]\partial_\mu\Psi \\ &= \int \bar{\Psi}:(i\hat{H} - \omega_\mu\hat{\gamma}^\mu)\hat{\mathbf{e}}[\widehat{d\omega}]:\Psi - \int \bar{\Psi}:[\widehat{d\omega}]\hat{\mathbf{e}}(i\hat{H} + \gamma^\mu\omega_\mu):\Psi \\ &= \dots = \langle:\hat{\mathbf{e}}\bar{\omega}_\mu\hat{\gamma}^\mu + \hat{\mathbf{e}}\hat{\gamma}^\mu\omega_\mu:\rangle, \end{aligned} \quad (22)$$

is equivalent to the fulfillment on average of the property (8) of curved-space Dirac matrices and spin connection, because the left hand side of (22) is  $\partial_\mu\langle\hat{\mathbf{e}}\hat{\gamma}^\mu\rangle$ . As eq. (8) is a consequence of the first of the Einstein-Palatini equations in (15), the result in (22) can be seen as a first indication that our precanonical quantization is consistent in the classical limit at least with the classical geometry underlying GR.

Note, however, that we had to omit several essential intermediate details in the calculation of (22). The most important one is that the terms with  $\Lambda$  do not cancel on their own, that would lead to the violation of the covariant probability conservation law due to the cosmological constant. Though it might sound plausible,

we think that the true message here is that the cosmological constant, which was introduced in the classical Lagrangian (11), should be cancelled by the constants generated by a proper choice of the ordering inside  $[\dots]$ . This cancellation yields a prediction of the admissible value of  $\Lambda$ , though in terms of an yet unspecified scale  $\varkappa$  introduced by precanonical quantization,<sup>12</sup> and fixes the ordering ambiguity in  $\hat{H}$  by requiring  $[\omega\omega\partial_\omega\partial_\omega]$  to be anti-Hermitean.

**Ehrenfest theorem for the Einstein's equations** can be obtained now for the DW Hamiltonian form of the latter (15). By proceeding similarly to (22), we obtain

$$\mathbf{d}_\alpha \langle \hat{\mathbf{p}}_{\omega_{\beta}^{IJ}}^\alpha \rangle = -i\hbar\varkappa\partial_\alpha \text{Tr} \int \widehat{\Psi} \hat{\mathbf{e}}^{\gamma\alpha} \partial_{\omega_{\beta}^{IJ}} [\widehat{d\omega}] \Psi = \dots = -\langle \partial_{\omega_{\beta}^{IJ}} \hat{\mathfrak{H}} \rangle, \quad (23)$$

that reproduces the first of the Einstein-Palatini equations, eq. (15a), on average.

The Einstein's equations proper, eq. (15b), are more tricky to obtain as the Ehrenfest-type statement. Using  $\mathfrak{C}_{e_K\omega_{\beta}^{IJ}}^\alpha \mathbf{d}_{[\alpha\omega_{\beta}^{IJ}]} = \mathbf{d}_\alpha (\mathfrak{C}_{e_K\omega_{\beta}^{IJ}}^\alpha \omega_{\beta}^{IJ}) - \mathbf{d}_\alpha (\mathfrak{C}_{e_K\omega_{\beta}^{IJ}}^\alpha) \omega_{\beta}^{IJ}$ , eq. (15a), the expression of  $\mathfrak{C}_{e\omega}$  in (13), and the constraint (12b), the Einstein's equations can be cast in the form more suitable for us:

$$\mathbf{d}_\alpha (\mathfrak{C}_{e_K\omega_{\beta}^{IJ}}^\alpha \omega_{\beta}^{IJ}) = \partial_{e_K} \mathfrak{H}. \quad (24)$$

The construction of the operators involved in (24) makes use of the observations that (i)  $\widehat{\partial_{e_K} \gamma} A = q[i\omega_{\gamma}^{KL} \bar{\gamma}_L, \hat{A}]$ , where  $q$  is a numerical/sign factor, (ii)  $\mathfrak{C}_{e_K\omega_{\beta}^{IJ}}^\alpha \omega_{\beta}^{IJ} \approx -\partial_{e_K} \gamma (\omega_{\beta}^{IJ} \mathbf{p}_{\omega_{\beta}^{IJ}}^\alpha(e))$ , and (iii) for any  $A^\alpha$ ,  $\mathbf{d}_\alpha A^\alpha = \mathbf{d}_\alpha(dx^\alpha \bullet A^\nu \varpi_\nu)$ .<sup>a</sup> Using the representations  $\widehat{dx^\nu \bullet} = (-1)^{n-1} \varkappa \hat{\gamma}^\mu$  (cf. Ref. 7) and  $\widehat{\mathbf{p}_{\omega}^\nu \varpi_\nu} = -i\hbar\partial_\omega$ , we get  $\widehat{\mathfrak{C}_{e_K\omega_{\beta}^{IJ}}^\alpha \omega_{\beta}^{IJ}} = -i\hbar q : \hat{\gamma}^\alpha \omega_{\gamma}^{KL} \bar{\gamma}_L :$ .

Now, using the precanonical Schrödinger equation (17) and its conjugate (18), we obtain

$$\mathbf{d}_\alpha \langle \widehat{\mathfrak{C}_{e_K\omega_{\beta}^{IJ}}^\alpha \omega_{\beta}^{IJ}} \rangle = \mathbf{d}_\alpha \text{Tr} \int \widehat{\Psi} \hat{\gamma}^\alpha q (-i\hbar) : \omega_{\gamma}^{KL} \bar{\gamma}_L [\widehat{d\omega}] : \Psi = \dots = \langle : \partial_{e_K} \gamma \mathfrak{H} : \rangle, \quad (25)$$

thus reproducing the Einstein's equations in the form (24) as the equation of expectation values of precanonical quantum operators.

## 4. Conclusion

The ability of precanonical quantization to reproduce correctly the classical field equations as the equations of expectation values of quantum operators, i.e. the validity of the Ehrenfest theorem, can be considered as a consistency test of the (i) precanonical representation of operators, (ii) precanonical analog of the Schrödinger equation and (iii) the prescription of the calculation of expectation values of operators using the Clifford algebra-valued precanonical wave functions. In the case

<sup>a</sup>  $A \bullet B := *^{-1}(*A \wedge *B)$  is the product operation in the Poisson-Gerstenhaber algebra of brackets which underlies the precanonical quantization.

of scalar fields on curved background the generalization of the Ehrenfest theorem identifies the connection term in the Dirac operator of the precanonical Schrödinger equation as the spin connection. This observation allows us to proceed with the precanonical quantization of general relativity with more confidence.

The sketch of the derivation of the Einstein-Palatini equations for vielbein gravity from the quantum theory of gravity obtained by precanonical quantization of vielbein general relativity indicates, in particular, that the value of the cosmological constant is fixed by a suitable ordering of operators and the consistency with the local covariant probability conservation, which also coincides with the known property of the spin connection and curved Dirac matrices fulfilled on average. A very rough estimation<sup>12c</sup> of the value of the dimensionless parameter  $\lambda = \frac{\Lambda}{(\hbar\kappa\kappa)^2} \sim n^3$ , which originates from the re-ordering of operators  $\omega$  and  $\partial_\omega$  in precanonical Schrödinger equation, has lead us to an unexpected conclusion that the parameter  $\kappa$  of precanonical quantization, which is consistent with the observed values of  $\Lambda$ ,  $G$  and  $\hbar$ , is at the nuclear scale. However, this preliminary consideration of the pure gravitational contribution to  $\Lambda$  neglects the matter fields. We need an independent argument regarding the estimation of the scale of  $\kappa$ , if it is a universal scale, which might be provided e.g. by a precanonical quantization approach to the mass gap problem in quantum Yang-Mills theory (see Ref. 16 for a work in this direction).

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## References

1. I. Kanatchikov, *Int. J. Theor. Phys.* **37**, 333 (1998). [arXiv:quant-ph/9712058](#).  
I. Kanatchikov, *AIP Conf. Proc.* **453**, 356 (1998). [arXiv:hep-th/9811016](#).  
I. Kanatchikov, *Rep. Math. Phys.* **43**, 157 (1999). [arXiv:hep-th/9810165](#).
2. I. Kanatchikov, [arXiv:hep-th/0112263](#); I. Kanatchikov, in *The Ninth Marcel Grossmann Meeting*, p. 1395, (World Scientific, Singapore, 2002). [arXiv:gr-qc/0012038](#).
3. H. Rund, *The Hamilton-Jacobi Theory in the Calculus of Variations*, (Robert E. Krieger, N.Y., 1973); H. Kastrup, *Phys. Rep.* **101**, 1 (1983).
4. I. Kanatchikov, *Rep. Math. Phys.* **41**, 49 (1998). [arXiv:hep-th/9709229](#).  
I. Kanatchikov, *Rep. Math. Phys.* **40**, 225 (1997). [arXiv:hep-th/9710069](#).  
I. Kanatchikov, [arXiv:hep-th/9312162](#).
5. B. Hatfield, *Quantum Field Theory of Point Particles and Strings*, (Addison-Wesley, Redwood City, 1992)
6. I. Kanatchikov, *Adv. Theor. Math. Phys.* **18**, 1249 (2014). [arXiv:1112.5801](#).  
I. Kanatchikov, [arXiv:1312.4518](#).



7. I. Kanatchikov, *J. Geom. Symmetry Phys.* **37**, 43 (2015). [arXiv:1501.00480](#).
8. I. Kanatchikov, *On the Ehrenfest Theorem in quantum general relativity*, work in progress.
9. I. Kanatchikov, *Precanonical Quantization: From Foundations to Quantum Gravity*, in preparation.
10. I. Kanatchikov, to appear in MG14 Proceedings (2016). [arXiv:1512.09137](#).
11. R. Bertlmann, *Anomalies in Quantum Field Theory*, (Clarendon, Oxford, 1996)
12. I. Kanatchikov, *AIP Conf. Proc.* **1514**, 73 (2012). [arXiv:1212.6963](#).  
 I. Kanatchikov, *J. Phys. Conf. Ser.* **442**, 012041 (2013). [arXiv:1302.2610](#).  
 I. Kanatchikov, *NPCS* **17**, 372 (2014). [arXiv:1407.3101](#).
13. I. Kanatchikov, *Int. J. Theor. Phys.* **40**, 1121 (2001). [gr-qc/0012074](#).  
 See also: I. Kanatchikov, [gr-qc/9810076](#); [gr-qc/9912094](#); [gr-qc/0004066](#).
14. I. Kanatchikov, In: *Differential Geometry and its Applications*, Proc. 10th Int. Conf. on Diff. Geom. & Appl., Olomouc, August 2007, Kowalski O., Krupka D., Krupková O. and Slovák J. (Eds), World Scientific, Singapore 2008, pp 615-625. [arXiv:0807.3127](#).
15. I. Kanatchikov, *Precanonical Quantum Cosmology and Cosmological Spin Connection Field*, talk at MG14.
16. I. Kanatchikov, *Rep. Math. Phys.*, **53**, 181 (2004). [arXiv:hep-th/0301001](#).